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UNIFORMLY SEPARATING FAMILIES OF FUNCTIONS

BY

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ABSTRACT

A concept concerning separation of points by functions is defined and studied. This concept has close relations with superpositions of functions and dimension theory, and these relations are investigated. A theorem concerning the dimension of projections of Cantor manifolds in R^m is proved.

1. Introduction

This article is devoted to the investigation of a concept concerning separation of points by functions, which we call uniform separation. It turns out that this concept has close relations with superpositions of functions and dimension theory. Our main emphasis will be on the investigation of the connection between these concepts.

Let X be a set, and let F be a family of functions on X. (We disregard the range of elements of F for the time being.) Recall that F is called a (point) separating family if for each $x \neq z$ in X, $\varphi(x) \neq \varphi(z)$ for some $\varphi \in F$. The property of uniform separation is stronger than that of separation.

1.1. Definition. (i) We call F a *strongly separating family* if for each integer m and each pair $\{x_i\}_{i=1}^m$, $\{z_i\}_{i=1}^m$ of disjoint m tuples of points in X there is some $\varphi \in F$ so that the sets $\{\varphi(x_i)\}_{i=1}^m$, $\{\varphi(z_i)\}_{i=1}^m$ do not coincide.

(ii) F is called a *uniformly separating family* (u.s.f.) if there exists a $0 < \lambda \leqq 1$ such that for each pair $\{x_i\}_{i=1}^m$, $\{z_i\}_{i=1}^m$ of disjoint finite sequences in X, there exists some $\varphi \in F$ so that if from the two sequences $\{\varphi(x_i)\}_{i=1}^m$ and $\{\varphi(z_i)\}_{i=1}^m$ in $\varphi[X]$ we remove a maximal number of pairs of points $\varphi(x_{i_1})$ and $\varphi(z_{i_2})$ with $\varphi(x_{i_1}) = \varphi(z_{i_2})$ there remains at least λm points in each sequence. (Or, equivalently, at most $(1-\lambda)m$ pairs can be removed; see also Definition 2.2.)

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Clearly if F is a u.s.f. then F is strongly separating, and a strongly separating family separates points. (Just take $m = 1$ in the above definition.) The following examples show that the converses are false.

EXAMPLES. (i) Let $X = \{(x, y): 0 \leqq x, y \leqq 1\}$ be the unit square. Let $F = \{\varphi_1, \varphi_2\}$ where $\varphi_1(x, y) = x$ and $\varphi_2(x, y) = y$. Clearly F separates points, but is not strongly separating: Let $x_1 = (0,0)$, $x_2 = (1,1)$, $z_1 = (1,0)$ and $z_2 = (0,1)$. Then $\{x_1, x_2\} \cap \{z_1, z_2\} = \emptyset$ but $\varphi_i[\{x_1, x_2\}] = \varphi_i[\{z_1, z_2\}] = \{0,1\}$, $i = 1,2$. Hence F is not a strongly separating family. (Throughout this paper we use the symbol $\varphi[U]$ for the image of the *set* U under the function φ, and $\varphi(x)$ for the image of the *point* x under φ.)

(ii) Let $X = N$ be the natural numbers, and let $\varphi_1(n) = [n/2]$ and $\varphi_2(n) = [(n+1)/2]$. Then $F = \{\varphi_1, \varphi_2\}$ is strongly separating, but not u.s.f. To show that F is strongly separating, let $A = \{n_1, n_2, \cdots, n_m\}$, $B = \{k_1, k_2, \cdots, k_m\}$ be disjoint sequences in N. The assumption that $\varphi_i[A] = \varphi_i[B]$, $i = 1,2$ would imply that for each $n_l \in A$, $n_l \pm 1 \in B$, and vice versa, which is clearly impossible. Hence F is strongly separating. But F is not a u.s.f., indeed let $A_n = \{1, 3, \cdots, 2n-1\}$, $B_n = \{2, 4, \cdots, 2n\}$. Then $A_n \cap B_n = \emptyset$, $|A_n| = |B_n| = n$ ($|A|$ denotes the cardinality of A) and

$$\varphi_1[A_n] = \{0, 1, 2, \cdots, n-1\}, \qquad \varphi_1[B_n] = \{1, 2, \cdots, n\}$$

$$\varphi_2[A_n] = \{1, 2, \cdots, n\}, \qquad \varphi_2[B_n] = \{1, 2, \cdots, n\}.$$

Hence $|\varphi_i[A_n] \cap \varphi_i[B_n]| \geqq n-2$, $i = 1,2$ and F is not a u.s.f.

(iii) Let X be the boundary of the triangle in I^2 with vertices $(0,0)$, $(\frac{1}{2}, 0)$ and $(1,1)$, and let φ_1, φ_2 and F be defined as in (i) above. It is not hard to check that F is strongly separating but not a u.s.f.

In Section 2 we show that for a finite family $F = \{\varphi_i\}_{i=1}^k$ of functions on X, being a u.s.f. is equivalent to the representability of each real bounded function f on X as $f(x) = \sum_{i=1}^k g_i(\varphi_i(x))$ where the g_i's are real bounded functions on the range Y_i of φ_i, $1 \leqq i \leqq k$, and that in the case where X is a compact metric space, and the elements of F are continuous functions on X, the representability of each continuous real function f on X in the above form with g_i continuous on Y_i implies that F is a u.s.f.

Kolmogorov [7] and Ostrand [11] proved theorems from which, in particular, it follows that for each compact n-dimensional metric space X, there exist continuous real functions $\{\varphi_i\}_{i=1}^{2n+1}$ such that each continuous real function f on X can be represented as $f(x) = \sum_{i=1}^{2n+1} g_i(\varphi_i(x))$ where the g_i's are real continuous functions on the real line.

Hence, it follows that each n-dimensional compact metric space admits a u.s.f. consisting of $2n + 1$ real valued continuous functions.

We conjecture that this number $2n + 1$ is the best possible, i.e., that the following holds:

1.2. CONJECTURE. Let X be a compact n-dimensional metric space ($n \geqq 2$) then no family consisting of $2n$ real continuous functions on X is a u.s.f.

The main part of the paper is devoted to results related to this conjecture. In particular we prove the conjecture for $n = 2, 3, 4$.

In Section 3 we state a lemma which gives sufficient conditions for a finite family of $2n$ continuous real functions on a compact metric space in order not to be a u.s.f. This lemma will be our main tool in proving Conjecture 1.2 for $n = 2, 3, 4$, and we hope that it will be useful in proving it for all $n \geqq 2$. The lemma will be proved in Section 6.

In Section 4 we state and prove a theorem concerning the dimension of projections of Cantor manifolds in R^m. This theorem has no direct connection with u.s.f., but will be used in proving Conjecture 1.2. In that section we also recall some theorems on closed mappings which lower dimension that will be needed later.

In Section 5, finally, we prove Conjecture 1.2 for $n = 2, 3, 4$.

Some remarks concerning Conjecture 1.2 are in place here; first the restriction $n \geqq 2$ in Conjecture 1.2 cannot be omitted: if X is an interval, the function $\varphi(x) = x$ clearly separates points uniformly. (For a family consisting of one function the concepts of separation and uniform separation coincide.) It is also clear that a single continuous real function separates the points of a space X if and only if X is topologically contained in the real line.

In [12, §3] it has been proved that a family consisting of two real continuous functions on the circle is not a u.s.f. there. It follows that if X is a one-dimensional space which contains a circle then Conjecture 1.2 holds for X. Hence, the only one-dimensional spaces on which we do not have full information about the minimal number of real continuous functions which separates their points uniformly are these spaces which do not contain a circle. Such spaces which are also connected and locally connected are called trees (or dendrites). Superpositions of functions on trees were studied intensively in Arnold's paper [1], and played a central role in solving Hilbert's problem 13, and in the development of the theory of superpositions of functions.

Conjecture 1.2, if true, combined with Ostrand's result [11], and our observations in Section 2, would imply the following characterization of dimension for compact metric spaces: $\dim X \leqq n$ if and only if there exists a u.s.f. for X consisting of $2n+1$ real continuous functions ($n>0$). In view of Section 5, this characterization holds for $n \leqq 4$ at least. Let us recall that from the classical theorem of Menger and Nöbeling (see, e.g., [4]) it follows that if $\dim X \leqq n$, then there exist $2n+1$ real continuous functions on X which separate its points. The number $2n+1$ here is minimal in the sense that for each n there exists a space X_n of dimension n such that X_n cannot be separated by $2n$ real continuous functions (see Flores [3]). But the number $2n+1$ is obviously not minimal for *each* n-dimensional space. I^n for example can be separated by n real continuous functions. Moreover, for each $n \leqq k \leqq 2n+1$ there exists a space X_n^k of dimension n so that the minimal number of real continuous functions which separate its points is k. It follows that the minimal number $n \leqq k \leqq 2n+1$ of real continuous functions which separate the points of an n-dimensional space X depends on the *global* combinatorial structure of X.

The situation with *uniform* separation is different (for $n \geqq 2$). Here the dimension of X (which is a local property) is the only factor which influences the minimal number (as we prove for $n = 2,3,4$, and conjecture for all $n \geqq 2$).

2. Uniformly separating families of functions and their connection with superpositions of functions and dimension theory

We begin this section with another definition of a u.s.f., more formal than the one given in the introduction. For this definition we need the concept of a set with multiplicity.

2.1. Definition. A set with multiplicity is a pair (A, a) where A is a set, and a is a function on A whose values are nonnegative integers. (Intuitively $a(x)$ is the number of times $x \in A$ appears in (A, a).)

The cardinality $|(A,a)|$ of the set with multiplicity (A,a) is defined by $|(A,a)| = \Sigma_{x \in A}\, a(x)$, and the intersection of two sets with multiplicity (A,a) and (B,b) is the set with multiplicity $(A \cap B, a \wedge b)$, where $(a \wedge b)(x) = \min\{a(x), b(x)\}$. If (A,a) is a set with multiplicity, and φ is a function on A, then $\varphi[(A,a)]$ is the set with multiplicity $(\varphi[A], \varphi a)$, where $\varphi a(y)$ is defined by

$$\varphi a(y) = \sum_{x \in A \cap \varphi^{-1}(y)} a(x) \quad \text{for } y \in \varphi[A].$$

Having this definition in mind we can define u.s.f. as follows:

2.2. Definition. Let X be a set, and let F be a family of functions on X. We call F a u.s.f. if there exists a constant $0 < \lambda \leqq 1$ such that for each pair of sets with multiplicity (A, a) and (B, b) in X with $A \cap B = \emptyset$ and $|(A, a)| = |(B, b)| = m < \infty$ there exists a $\varphi \in F$ so that $|\varphi[(A, a)] \cap \varphi[(B, b)]| \leqq (1-\lambda)m$.

It is easy to check that Definitions 1.1 and 2.2 agree.

Let X be a set. We denote by $l_1(X)$ the Banach space of real functions μ on X such that $\|\mu\| = \Sigma_{x \in X} |\mu(x)| < \infty$ (i.e. $\{x : \mu(x) \neq 0\}$ is a countable set $\{x_n\}_{n=1}^{\infty}$ and $\Sigma_{n=1}^{\infty} |\mu(x_n)| < \infty$; for convenience we shall use the symbol $\mu = \Sigma_{n=1}^{\infty} a_n \delta_{x_n}$ for $\mu \in l_1(X)$ where $a_n = \mu(x_n)$). $B(X)$ will denote the Banach space of bounded real functions on X with the norm $\|f\| = \sup_{x \in X} |f(x)|$. It is well known that $(l_1(X))^* = B(X)$.

If X is a compact metric space, then $C(X)$ will denote the closed subspace of $B(X)$ consisting of the continuous functions. We identify $C(X)^*$ with the space of regular Borel measures on X with the total variation as the norm. Observe that $l_1(X)$ is the closed subspace of $C(X)^*$ spanned by the singletons δ_x, $x \in X$.

All the notations and theorems concerning Banach spaces, linear operators, and their adjoints which we use in the article can be found in [2].

Let $F = \{\varphi_i\}_{i=1}^{k}$ be a finite family of functions on X such that φ_i maps X onto the set Y_i, $1 \leqq i \leqq k$, and let $Y = \bigcup_{i=1}^{k} Y_i$ be the (disjoint) union of the Y_i's.

Consider the following operators:

$$T_i : l_1(X) \to l_1(Y_i) \qquad 1 \leqq i \leqq k \tag{2.3}$$

defined by

$$T_i\mu(y) = \sum_{x \in \varphi_i^{-1}(y)} \mu(x); \qquad \mu \in l_1(X), y \in Y_i,$$

or, equivalently, if $\mu = \Sigma_{n=1}^{\infty} a_n \delta_{x_n}$ then

$$T_i\left(\sum_{n=1}^{\infty} a_n \delta_{x_n}\right) = \sum_{n=1}^{\infty} a_n \delta_{\varphi_i(x_n)};$$

and

$$T : l_1(X) \to l_1(Y) \tag{2.4}$$

defined by

$$T = \sum_{i=1}^{k} T_i$$

i.e.

$$T\left(\sum_{n=1}^{\infty} a_n \delta_{x_n}\right) = \sum_{i=1}^{k} T_i\left(\sum_{n=1}^{\infty} a_n \delta_{x_n}\right) = \sum_{i=1}^{k} \sum_{n=1}^{\infty} a_n \delta_{\varphi_i(x_n)}.$$

REMARK. We regard $l_1(Y_i)$, $1 \leqq i \leqq k$ as the subspace of $l_1(Y)$ consisting of the functions which vanish outside Y_i. The same convention applies to $B(Y_i)$, $C(Y_i)$ and $C(Y_i)^*$.

We define further operators $U_i : B(Y_i) \to B(X)$ and $U : B(Y) \to B(X)$ by

(2.5) $$U_i g(x) = g(\varphi_i(x)); \qquad g \in B(Y_i),\ x \in X$$

and

(2.6) $$Ug(x) = \sum_{i=1}^{k} g(\varphi_i(x)); \qquad g \in B(Y),\ x \in X.$$

2.7. PROPOSITION. *The operators U_i, T_i, $1 \leqq i \leqq k$, U and T are linear and bounded. Moreover $T_i^* = U_i$ for $1 \leqq i \leqq k$ and $T^* = U$.*

PROOF. Trivial. Observe, e.g., that for $x \in X$ and $g \in B(Y_i)$

$$(T_i^*(g)(x) = (T_i^* g)(\delta_x) = (T_i \delta_x) g = (\delta_{\varphi_i(x)}) g = g(\varphi_i(x)).$$

Hence by (2.5) $T_i^* = U_i$. □

The following theorem links the concepts of u.s.f. with superpositions of functions:

2.8 THEOREM. *In our previous notation, the following three statements are equivalent*:

(i) *F is a u.s.f.*

(ii) *Each $f \in B(X)$ is representable in the form*

$$f(x) = \sum_{i=1}^{k} g_i(\varphi_i(x)) \quad \textit{with} \quad g_i \in B(Y_i).$$

(iii) *The operator T (defined in (2.4)) is an isomorphism into.*

PROOF. Clearly (ii) means that the operator U maps $B(Y)$ onto $B(X)$. It is well known that a bounded linear operator T on a Banach space is an isomorphism into if and only if its adjoint T^* maps onto. Since by Proposition 2.7 $T^* = U$, we get that (ii) $\Leftrightarrow$ (iii).

Assume that (iii) holds. It follows that there exists a $\beta > 0$ such that $\| T\mu \| \geqq \beta \| \mu \|$ for all $\mu \in l_1(X)$; hence $\| T_i \mu \| \geqq (\beta / k) \| \mu \|$ for some $1 \leqq i \leqq k$. Notice that $\| T_i \mu \| \leqq \| \mu \|$ and therefore $\beta \leqq k$.

To show that F is a u.s.f. let (A, a) and (B, b) be disjoint sets with multiplicity in X, with

$$|(A, a)| = |(B, b)| = m.$$

Consider the element $\mu \in l_1(X)$ defined by

$$\mu = \sum_{x \in A} a(x)\delta_x - \sum_{x \in B} b(x)\delta_x. \tag{2.9}$$

Clearly $\|\mu\| = 2m$. From Definition 2.1 and (2.3) it follows that

$$\| T_i\mu \| = 2m - 2|\varphi_i[(A,a)] \cap \varphi_i[(B,b)]|.$$

As we observed $\| T_i\mu \| \geqq (\beta/k)\|\mu\|$ for some $1 \leqq i \leqq k$, i.e. for this i, $|\varphi_i[(A,a)] \cap \varphi_i[(B,b)]| \leqq (1-\beta/k)m$. Hence F is u.s.f. with the constant $\lambda = \beta/k$. This proves (iii) $\Rightarrow$ (i).

We still have to show that (i) $\Rightarrow$ (ii). Assume F is u.s.f. with the constant $\lambda > 0$. It follows (as in the proof of (iii) $\Rightarrow$ (i)) that for all $\mu \in l_1(X)$ of the form (2.9) $\| T\mu \| \geqq \lambda \|\mu\|$. We wish to show that $U = T^*$ maps onto $B(X)$. The constant functions are clearly in the range of U, hence we may consider $B(X)$ modulo the constants. The predual of this last space is

$$l_1(X)_0 = \{\mu \in l_1(X) : \sum_{x \in X} \mu(x) = 0\}.$$

Hence it suffices to show that $\| T\mu \| \geqq \lambda \|\mu\|$ for all $\mu \in l_1(X)_0$. The elements with finite support, and rational values are norm dense in $l_1(X)_0$, thus we may consider such ones. Let $\mu = \Sigma_{l=1}^L r_l \delta_{x_l}$ be such an element, with $\|\mu\| = \Sigma_{l=1}^L |r_l| = 1$. We may assume that $r_l = n_l/2m$ where $m > 0$, n_l, $1 \leqq l \leqq L$, are integers, and $2m$ is the common denominator of the r_l's. Let us also assume that $n_1, \cdots, n_s$ are positive, and that $n_{s+1}, \cdots, n_L$ are negative. Since $\Sigma_{l=1}^L r_l = 0$ we clearly have $\Sigma_{l=1}^s n_l = \Sigma_{l=s+1}^L |n_l| = m$. Consider the sets with multiplicity (A,a) and (B,b) defined in the following manner:

$$A = \{x_l\}_{l=1}^s, \qquad a(x_l) = n_l \qquad (\text{for } x_l \in A)$$

$$B = \{x_l\}_{l=s+1}^L, \qquad b(x_l) = -n_l \qquad (\text{for } x_l \in B).$$

Then $\mu = (1/2m)[\Sigma_{x \in A} a(x)\delta_x - \Sigma_{x \in B} b(x)\delta_x]$, i.e., $2m\mu$ is of the form (2.9). Hence, as we observed, $\| T\mu \| \geqq \lambda \|\mu\|$, and the theorem follows. □

Assume now that X is a compact metric space, and that F consists of continuous functions $\{\varphi_i\}_{i=1}^k$. Hence the Y_i's, $1 \leqq i \leqq k$ and Y are compact metric spaces too. Let $V_i : C(Y_i) \to C(X)$, $1 \leqq i \leqq k$ and $V : C(Y) \to C(X)$ be the restrictions of U_i, $1 \leqq i \leqq k$ and U to $C(Y_i)$ and $C(Y)$ respectively, i.e.

$$(V_i g)(x) = g(\varphi_i(x)); \qquad g \in C(Y_i),\ x \in X \tag{2.10}$$

and

$$(Vg)(x) = \sum_{i=1}^{k} g(\varphi_i(x)); \qquad g \in C(Y),\ x \in X. \tag{2.11}$$

As in the discrete case, we can derive information from the nature of the adjoints V_i^* and V^* of V_i and V:

2.12. Proposition. *Let $\mu \in C(X)^*$, $G_i \subset Y_i$ and $G \subset Y$ be Borel sets. Then*

$$(V_i^*\mu)(G_i) = \mu(\varphi_i^{-1}(G_i)) \tag{2.13}$$

and

$$(V^*\mu)(G) = \sum_{i=1}^{k} \mu(\varphi_i^{-1}(G)). \tag{2.14}$$

Proof. Routine.

We may consider the elements of $l_1(X)$ (resp. $l_1(Y_i)$) as Borel measures on X (resp. on Y_i), i.e. $l_1(X) \subset C(X)^*$. Observe also that by (2.3), (2.4) and Proposition 2.12 we have

$$V_i^*/l_1(X) = T_i \quad \text{and} \quad V^*/l_1(X) = T. \tag{2.15}$$

The next proposition, which links the concepts of u.s.f., superpositions with bounded functions and superpositions with continuous functions, follows at once.

2.16. Proposition. *Let X be a compact metric space, let $F = \{\varphi_i\}_{i=1}^{k}$ be a finite family of continuous functions on X mapping it onto the spaces $\{Y_i\}_{i=1}^{k}$, and let $Y = \bigcup_{i=1}^{k} Y_i$ (disjoint union).*

If each $f \in C(X)$ admits a representation

$$f(x) = \sum_{i=1}^{k} g_i(\varphi_i(x)) \quad \text{with} \quad g_i \in C(Y_i)$$

then each $f \in B(X)$ admits a representation

$$f(x) = \sum_{i=1}^{k} g_i(\varphi_i(x)) \quad \text{with} \quad g_i \in B(Y_i)$$

i.e. F is a u.s.f.

Proof. The assumption means that V maps $C(Y)$ onto $C(X)$; hence V^* is an isomorphism. Since $T = V^*/l_1(X)$, T is an isomorphism too, and the proposition follows from Theorem 2.8. □

We could not prove the converse of Proposition 2.16, i.e., that if F is a u.s.f. then V maps $C(Y)$ onto $C(X)$ except in the cases $k = |F| = 1$ (which is trivial) and $k = 2$.

Let us state this question as a problem:

2.17. PROBLEM. Is it true that if F is a u.s.f. then V maps $C(Y)$ onto $C(X)$?

In order to show that in the case $k=2$ the answer to Problem 2.17 is affirmative, we exhibit a sufficient condition for a finite family $F=\{\varphi_i\}_{i=1}^k$ to be a u.s.f. on a set X, which in the case $k=2$ turns out to be necessary.

Let X be a set, and let $F=\{\varphi_i\}_{i=1}^k$ be a family of functions on X. For a subset Z of X and $1\leqq i\leqq k$ define

$$Z^{(i)}=\{x\in Z:|Z\cap\varphi_i^{-1}(\varphi_i(x))|\geqq 2\}. \tag{2.18}$$

Let $\tau:2^X\to 2^X$ be the function defined by

$$\tau(Z)=\bigcap_{i=1}^{k} Z^{(i)} \tag{2.19}$$

(where 2^X denotes the set of all subsets of X).

2.20. THEOREM. *If $\tau^n(X)=\varnothing$ for some integer n then F is a u.s.f.*

More generally: Let Σ be a field of subsets of X so that $Z^{(i)}\in\Sigma$ for $Z\in\Sigma$ and $1\leqq i\leqq k$. If $\tau^n(X)=\varnothing$ for some n, then there exists a $0<\lambda\leqq 1$ so that for all real measures μ on (X,Σ) $\|W_i\mu\|\geqq\lambda\|\mu\|$ for some $1\leqq i\leqq k$, where $W_i\mu$ is the measure on $Y_i=\varphi_i[X]$ defined by $W_i\mu(G)=\mu(\varphi_i^{-1}(G))$.

REMARK. If X is compact metric and the φ_i's are continuous on X, then the Borel field has this property since

$$Z^{(i)}=\{x\in Z:|Z\cap\varphi_i^{-1}(\varphi_i(x))|\geqq 2\}=\{x\in Z;\ \text{diameter}\ (Z\cap\varphi_i^{-1}(\varphi_i(x)))>0\}$$

$$=\bigcup_{n=1}^{\infty}\{x\in Z:\text{diameter}\ (Z\cap\varphi_i^{-1}(\varphi_i(x)))\geqq \tfrac{1}{n}\}$$

i.e. $Z^{(i)}$ is an F_σ in Z.

It follows from (2.10) and (2.13) that in this case $\|V_i^*\mu\|\geqq\lambda\|\mu\|$ for all $\mu\in C(X)^*$. If we take Σ to be 2^X we see that F is a u.s.f.

PROOF. We use induction on n. If $\tau^0(X)=\varnothing$ the assertion vacuously holds. Assume that for $Z\in\Sigma$ $\tau^{n-1}(Z)=\varnothing$ implies that the assertion holds for Z, and that $\tau^n(X)=\varnothing$. Observe first that for each $1\leqq i\leqq k$, $\varphi_i[X^{(i)}]\cap\varphi_i[X\setminus X^{(i)}]=\varnothing$, since for $y_1\in\varphi_i[X^{(i)}]$, $|\varphi_i^{-1}(y_1)|\geqq 2$, while for $y_2\in\varphi_i[X\setminus X^{(i)}]$ $|\varphi_i^{-1}(y_2)|=1$.

Set $Z=\tau(X)$. We have: $\varnothing=\tau^n(X)=\tau^{n-1}(\tau(X))=\tau^{n-1}(Z)$. Hence by the induction hypothesis our assertion holds for Z with some constant $0<\lambda\leqq 1$. Let α be any real number $1/(1+\lambda)<\alpha<1$, e.g. $\alpha=(2+\lambda)/(2+2\lambda)$. Let μ be any real measure on Σ with $\|\mu\| = 1$ and denote by $|\mu|$ the variation of μ. Consider the following two cases:

(i) $|\mu|(Z) \geqq \alpha$. Thus $\|\mu/Z\| \geqq \alpha$, and since our assertion holds for Z, it follows that $\|W_i(\mu/Z)\| \geqq \lambda\alpha$. The total variation of $\mu/X\setminus Z$ does not exceed $1-\alpha$, hence the mass of μ which lays outside Z can reduce $\|W_i\mu\|$ by at most $(1-\alpha)$. It follows that $\|W_i\mu\| \geqq \lambda\alpha - (1-\alpha)$.

(ii) $|\mu|(Z) < \alpha$. Then $|\mu|(X\setminus Z) \geqq 1-\alpha$. Now $X\setminus Z = X\setminus \bigcap_{i=1}^k X^{(i)} = \bigcup_{i=1}^k (X\setminus X^{(i)})$. Hence $|\mu|(X\setminus X^{(i)}) \geqq (1-\alpha)/k$ for some $1 \leqq i \leqq k$. Since φ_i is one to one on $X\setminus X^{(i)}$ and $\varphi_i[X^{(i)}] \cap \varphi_i[X\setminus X^{(i)}] = \emptyset$, it follows that $\|W_i\mu\| \geqq (1-\alpha)/k$. Hence if we take $\lambda' = \lambda/2k(1+\lambda)$ then $\lambda' \leqq \min\{(1-\alpha)/k, \lambda\alpha - (1-\alpha)\}$ and our assertion holds with λ'. □

2.21. Lemma. *If $|F| = 2$ then F is a u.s.f. if and only if $\tau^n(X) = \emptyset$ for some n.*

Proof. Assume $\tau^n(X) \neq \emptyset$ for all $n \geqq 0$. Let $x_1 \in \tau^n(X) = (\tau^{n-1}(X))^{(1)} \cap (\tau^{n-1}(X))^{(2)}$. Since $x_1 \in (\tau^{n-1}(X))^{(1)}$ it follows from (2.18) that there exists some $x_2 \in \tau^{n-1}(X)$ with $\varphi_1(x_1) = \varphi_1(x_2)$. By the same argument there exists an $x_3 \in \tau^{n-2}(X)$ with $\varphi_2(x_2) = \varphi_2(x_3)$. In this way we construct points $\{x_j\}_{j=1}^n$ in X ($x_j \in \tau^{n-j+1}(X)$) with $\varphi_1(x_j) = \varphi_1(x_{j+1})$ for j odd, and $\varphi_2(x_j) = \varphi_2(x_{j+1})$ for j even.

Set $\mu = \sum_{j=1}^n (-1)^j \delta_{x_j}$. Then $\|\mu\| = n$ and clearly $\|T_i\mu\| \leqq 2$, $i = 1,2$. Hence $\|T_i\mu\| \leqq (2/n)\|\mu\|$, and since n was arbitrary F is not a u.s.f.

The other direction has been proved in Theorem 2.20. □

Let now X be a compact metric space and let $F = \{\varphi_1, \varphi_2\}$ be a u.s.f. on X with continuous elements. By Lemma 2.21 $\tau^n(X) = \emptyset$ for some n, hence, by Theorem 2.20 (and the remark following it) the operator V^* is an isomorphism into, which shows that the answer to (2.19) is affirmative in this case.

Let $F = \{\varphi_i\}_{i=1}^k$ be a family of continuous real valued functions on a compact metric space X. The Stone–Wierstrass theorem states that F separates points if and only if the closed algebra generated by F is $C(X)$.

An affirmative answer to Problem 2.17, combined with Proposition 2.16 would imply the following Stone–Wierstrass type theorem: F is a u.s.f. if and only if $C(X)$ is the sum of the closed algebras each of which is generated by a single element of F.

In [11] Ostrand proved the following:

2.22. Theorem. *Let $X = X_1 \times X_2 \times \cdots \times X_m$ where X_i are compact metric spaces with $\dim X_i = n_i$, $1 \leqq i \leqq m$ and $n = \sum_{i=1}^m n_i$. Then there exist functions $\varphi_1^i, \cdots, \varphi_{2n+1}^i$ in $C(X_i)$ such that each $f \in C(X)$ is representable as*

$$f(x_1, \cdots, x_m) = \sum_{j=1}^{2n+1} g_j[\varphi_j^1(x^1) + \varphi_j^2(x^2) + \cdots + \varphi_j^m(x^m)]$$

with $g_j \in C(R)$.

Combining a characterization of dimension presented in [11] with a technique from [12], one can prove that such families $\{\varphi_j^i\}$, $1 \leqq j \leqq 2n+1$, $1 \leqq i \leqq m$ are residual in $[\Pi_{i=1}^m C(X_i)]^{2n+1}$ (i.e. are complemented there by a set of first category).

By Proposition 2.16, Ostrands theorem implies the following (just set $i = 1$ and $\dim X = n = n_1$):

2.23. THEOREM. *Let X be an n-dimensional compact metric space. Then there exists a u.s.f. $F \subset C(X)$ with $|F| \leqq 2n+1$. Moreover, such families are residual in $C(X)^{2n+1}$.*

As stated in the introduction it seems that the number $2n+1$ in Theorem 2.23 is the best possible (see Conjecture 1.2). In the next sections we prove this for $n = 2, 3, 4$.

2.24. THEOREM. *Let X be a compact n-dimensional metric space ($n = 2, 3, 4$). Then no family $F \subset C(X)$ with $|F| \leqq 2n$ is a u.s.f.*

3. The main lemma

In this section we state a lemma which gives sufficient conditions on a family F consisting of $2n$ real continuous functions on a compact metric space to be non-u.s.f. We shall apply the lemma in the proof of Theorem 2.24 in Section 5. Since the proof of the lemma is technical and involved, we devote a special section to it (Section 6) at the end of the paper.

In order to state the main lemma we need some definitions and notation.

3.1. DEFINITION. Let X and Y be metric spaces, and let $\varphi : X \to Y$ be a continuous function. φ will be called *interior* if for each open $U \subset X$, $\varphi[U]$ has non-empty interior in Y. (Observe that an interior function φ need not be an open function, i.e. we do not demand that $\varphi[U]$ will be open for open U; $\varphi(t) = t^2$ on $[-1, 1]$ is interior but not open.)

3.2. DEFINITION. For an integer k, let $[k]$ denote the set $\{1, 2, \cdots, k\}$. a, b, c, will denote subsets of $[k]$, and $|a|$ the cardinality of a.

3.3. DEFINITION. Let X be a compact metric space, let $F = \{\varphi_i\}_{i=1}^k \subset C(X)$ and let $a \subset [k]$. We denote by φ_a the function

$$\varphi_a : X \to R^{|a|} = \prod_{i \in a} R_i$$

defined by

$$\varphi_a(x) = (\varphi_{i_1}(x), \varphi_{i_2}(x), \cdots, \varphi_{i_{|a|}}(x)); \qquad x \in X$$

where $a = \{i_1, \cdots, i_{|a|}\}$.

REMARK. Observe that in Definition 3.1 of an interior function it is important to specify the range space Y into which φ maps X. In the sequal we shall use this concept for the functions φ_a defined in Definition 3.3 and it is worth noting that these mappings were defined as mappings from X to $R^{|a|}$. Hence when we say that φ_a is interior, we mean interior as a mapping from X to $R^{|a|}$ (and not, e.g., as a mapping from X to $\varphi_a[X]$).

3.4. DEFINITION. Let X be a compact metric space, let $F = \{\varphi_i\}_{i=1}^{2n}$ be a family of $2n$ elements of $C(X)$, and let $a, b, c \subset [k]$.

(i) The triple (a, b, c) will be called *an interior triple w.r.t.* F if the following conditions hold:

1. The sets a, b, c are mutually disjoint.
2. $|a| + |b| = |a| + |c| = n$.
3. The functions $\varphi_{a \cup b}$ and $\varphi_{a \cup c}$ are interior.

(ii) A sequence $\{(a_i, b_i, c_i)\}_{i=1}^{k}$ of interior triples w.r.t. F is called *an interior chain w.r.t.* F if for each $1 \leqq i \leqq k-1$, $[2n] \setminus (a_i \cup b_i \cup c_i) = a_{i+1}$.

3.5. MAIN LEMMA. *Let X be a compact metric space, and let $F = \{\varphi_i\}_{i=1}^{2n} \subset C(X)$. If for each integer $k > 0$ there exists an interior chain $\{(a_i, b_i, c_i)\}_{i=1}^{k}$ of length k w.r.t F, then F is not a u.s.f.*

3.6. REMARK. In the proof of 3.5 to be presented in Section 6, we will obtain more precise information: we will see there that if there exists an interior chain $\{(a_i, b_i, c_i)\}_{i=1}^{k}$ w.r.t. F, then there exist k^2 points $\{x_i^j\}_{1 \leqq i,j \leqq k}$ in X such that the element $\mu \in l_1(X)$ defined by $\mu = \Sigma_{i,j}(-1)^{i+j}\delta_{x_i^j}$ has the property $\| T_l \mu \| < 8k$ for $1 \leqq l \leqq 2n$ (where T_l is the operator defined in (2.3)). Since $\| \mu \| = k^2$ and $\| T\mu \| = \Sigma_{l=1}^{2n} \| T_l \mu \|$ it follows that $\| T\mu \| < 2n \cdot 8k = (16n/k)\| \mu \|$. If this holds for arbitrary k, then T is not an isomorphism, and by Theorem 2.8 F is not a u.s.f.

4. On the dimension of projections of Cantor manifolds in R^m

The main aim of this section is to state and prove a theorem concerning the title of the section. This theorem generalizes a theorem of S. Mardešič [9] (see also [10]), and in the proof we use the technique established there. We also recall some known theorems on mappings which lower dimension.

Let $m > 0$ be an integer. We set $R^m = R_1 \times R_2 \times \cdots \times R_m$ where R_i is the real line. We use again $[m]$ for $\{1, 2, \cdots, m\}$.

4.1. DEFINITION. Let a be a subset of $[m]$. The projection P_a from R^m to $R^{|a|}$ is the mapping

$$P_a(x_1, x_2, \cdots, x_m) = (x_{i1}, x_{i2}, \cdots, x_{i|a|}) \qquad i_j \in a.$$

If a subset W of R^m is understood, we may use P_a instead of P_a/W.

4.2. LEMMA. *Let X be a compact subset of R^n, and let $1 \leqq k \leqq n$ be an integer. Let Y be the subset of R^k defined by*

$$(4.3) \qquad Y = \{y \in R^k : \dim[(y \times R_{k+1} \times R_{k+2} \times \cdots \times R_n) \cap X] \geqq n - k\}.$$

If Y is of second category in R^k then $\dim X \geqq n$.

4.4. REMARK. It is not hard to show that Y is an F_σ in R^k (see [5] or [6]). Thus the three properties

(i) Y is of second category in R^k,

(ii) Y has non-empty interior in R^k,

(iii) $\dim Y = k$

are equivalent.

Hence one could state the lemma as:

$$(4.5) \qquad \dim Y = k \text{ if and only if } \dim X = n.$$

(Clearly $\dim X = n$ implies $\dim Y = k$.)

PROOF. Let $\{B_l\}_{l=1}^{\infty}$ be a sequence of $(n-k)$-dimensional cubes in $R_{k+1} \times \cdots \times R_n$, which forms a basis for the topology of this space. For each $l \geqq 1$ set

$$(4.6) \qquad S_l = \{y \in R^k : y \times B_l \subset X\}$$

and

$$(4.7) \qquad S = \bigcup_{l=1}^{\infty} S_l.$$

We claim that $Y \subset S$.

Indeed, let $y \in Y$; then $\dim[(y \times R_{k+1} \times \cdots \times R_n) \cap X] = n - k$ hence $(y \times R_{k+1} \times \cdots \times R_n) \cap X$ has non-empty interior in $y \times R_{k+1} \times \cdots \times R_n$, i.e.

$(y \times R_{k+1} \times \cdots \times R_n) \cap X \supset y \times B_l$ for some $l \geqq 1$. Then $y \times B_l \subset X$, and by (4.6) $y \in S_l \subset S$.

Now, since Y is of second category, so is S. Hence there exists some $l \geqq 1$ so that $\bar{S}_l$ contains some k cube D. Since $y \times B_l \subset X$ for each $y \in S_l$, we also have $S_l \times B_l \subset X$, and by the compactness of X, $\overline{S_l \times B_l} \subset X$ too. Hence $D \times B_l \subset X$, and $\dim[D \times B_l] = k + (n-k) = n$. This proves the lemma. □

In his paper [9] S. Mardešič proved the following theorem (see also [10]):

4.8. Theorem. *Let W be an n-dimensional compact subset of R^m ($n \leqq m$). Then there exists a subset a of $[m]$ with $|a| = n$ so that* $\dim P_a[W] = n$.

Our next theorem is in the same direction. Let us recall that for a topological space W, $dc W \geqq n$ means that no closed subset of dimension $\leqq n-2$ disconnects W. (See [8] for more information on dc.)

A compact space W is called an n-dimensional Cantor manifold if $\dim W = dc W = n$.

Each n-dimensional compact metric space contains an n-dimensional Cantor manifold (see [4, p. 93]).

4.9. Theorem. *Let $W \subset R^m$ be compact, with $dc W \geqq n$; $n \leqq m$. (In particular, W may be an n-dimensional Cantor manifold.) If* $\dim P_i[W] = 1$ *for some $1 \leqq i \leqq m$ then there exists a subset a of $[m]$ with $|a| = n-1$, and $i \notin a$ so that* $\dim P_{\{i\} \cup a}[W] = n$.

Proof. Since W is connected and $\dim P_i[W] = 1$ we get that $P_i[W]$ is a closed interval $[\alpha, \beta]$, in R_i.

Let $\alpha < t < \beta$; since t disconnects $P_i[W]$, $P_i^{-1}(t)$ disconnects W, and since $dc W \geqq n$ it follows that $\dim P_i^{-1}(t) \geqq n-1$. By Theorem 4.8 there exists a subset a of $[m] \setminus \{i\}$ with $|a| = n-1$ so that $\dim P_a[P_i^{-1}(t)] = n-1$.

For $a \subset [m] \setminus \{i\}$ with $|a| = n-1$ set

$$T(a) = \{t : \alpha < t < \beta, \dim P_a[P_i^{-1}(t)] = n-1\}. \tag{4.10}$$

It follows that $\bigcup_a T(a) \supset \{t : \alpha < t < \beta\}$, where the union is taken over all subsets a of $[m] \setminus \{i\}$ with $|a| = n-1$.

Hence there exists some a such that $T(a)$ is of second category in (α, β). Set $X = P_{\{i\} \cup a}[W]$. By applying Lemma 4.2 with $k = 1$ and $Y = T(a)$ we get that $\dim P_{\{i\} \cup a}[W] = n$. □

Let us remark that the conditions of Theorem 4.9 cannot be weakened by assuming only that $\dim W \geqq n$, or even $\dim_w W \geqq n$ for all $w \in W$ instead of $dc W \geqq n$.

The following example, of a compact subset W of R^3 with $\dim_w W = 2$ for all $w \in W$, and $\dim P_1[W] = \dim P_{12}[W] = \dim P_{13}[W] = 1$, illustrates this fact.

4.11. Example. Set

$$I = \{(x, y, z) : 0 \leqq x \leqq 1; y = z = 0\} \subset R^3 \tag{4.12}$$

and for each dyadic number $k . 2^{-n}$ with k odd set

$$W_{k.2^{-n}} = \{(x, y, z) : x = k . 2^{-n}; 0 \leqq y \leqq 2^{-n}; 0 \leqq z \leqq 2^{-n}\} \tag{4.13}$$

and

$$W = \bigcup W_{k.2^{-n}} \bigcup I. \tag{4.14}$$

One can easily verify in Fig. 1 that W has the desired properties.

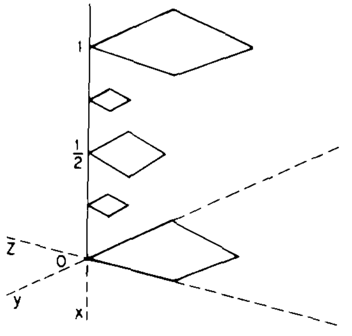

Fig. 1

There is a natural extension of Theorem 4.9, which we could neither prove nor disprove. Let us state it as a problem.

4.15. Problem. Let $W \subset R^m$ be compact, with $dc\, W \geqq n$, $n \leqq m$. If for some $k \leqq n$ $\dim P_{[k]}[W] = k$, does it follow that there exists an $a \subset [m] \setminus [k]$ with $|a| = n - k$ so that $\dim P_{[k] \cup a}[W] = n$?

Observe that Theorem 4.9 is the case $k = 1$. If Problem 4.15 has an affirmative answer then our proof of Theorem 2.24, which will be given in the next section, works for every $n \geqq 2$ (and not only $n = 2, 3, 4$).

We conclude this section by recalling some definitions and theorems concerning mappings which lower dimension of compact metric spaces. The theorems are valid for closed continuous mapping of general metric spaces, but we formulate them only for continuous mappings of compact spaces.

DEFINITION. Let X be compact metric, and f a continuous function on X. $\dim f$ is defined by

$$\dim f = \sup_{y \in f[x]} \dim f^{-1}(y). \tag{4.16}$$

The following is a well known theorem of Hurewicz ([4, p. 91]).

4.17. THEOREM. *Let X be a compact metric space, and let f be a continuous function on X. Then*

$$\dim X \leqq \dim f[X] + \dim f. \tag{4.18}$$

This theorem can be extended in two directions: first the trivial one.

4.19. THEOREM. $dc\, X \leqq dc\, f[X] + \dim f.$
(See [8] where it is stated without proof.)

PROOF. If C disconnects $f[X]$, then $f^{-1}(C)$ disconnects X. Since $C = f[f^{-1}(C)]$ we get by (4.18)

$$\dim C = \dim f[f^{-1}(C)] \geqq \dim f^{-1}(C) - \dim f, \tag{4.20}$$

i.e.

$$\dim f^{-1}(C) \leqq \dim C + \dim f. \tag{4.21}$$

Hence if some set of dimension n disconnects $f[X]$, then some set of dimension $\leqq n + \dim f$ disconnects X, and the theorem follows. □

Define

$$Df = \{y \in f[X] : \dim f^{-1}(y) \geqq \dim X - \dim f[X]\}. \tag{4.22}$$

Theorem 4.18 just states that $Df \neq \emptyset$. The following is a better estimate on $\dim Df$:

4.23. THEOREM (Jung–Keesling). $\dim X \leqq \dim Df + \dim f$. (See [5] and [6] for a proof.)

5. Proof of the non-existence of u.s.f. of cardinality $2n$ for n-dimensional spaces $n = 2, 3, 4$

The non-existence of u.s.f. of cardinality 4 for two-dimensional spaces has been established in [13], in a different setting, but with the same underlying ideas. Since by using the results of Sections 3 and 4 the proof becomes short, and for the sake of completeness, we repeat the proof here. We need first some lemmas and notations.

5.1. DEFINITION. Let X be an n-dimensional Cantor manifold and let $F \subset C(X)$ be a u.s.f. We call F a *minimal u.s.f.* if no subfamily of F is a u.s.f. on some n-dimensional Cantor manifold contained in X.

5.2. LEMMA. *Let X be an n-dimensional compact metric space, and let $F \subset C(X)$ be a finite u.s.f. Then there exists an n-dimensional Cantor manifold $X' \subset X$ and a subfamily $F' \subset F$ such that F' is a minimal u.s.f. on X'.*

We omit the simple proof.

5.3. LEMMA. *Let $F = \{\varphi_i\}_{i=1}^k$ be a u.s.f. on a set X. Let a, b be subsets of $[k]$ with $a \cap b = \emptyset$ and $a \cup b = [k]$. If φ_a is constant on a set $L \subset X$, and $Z \subset \varphi_b^{-1}(\varphi_b[L]) \setminus L$, then $\{\varphi_i\}_{i \in a}$ is a u.s.f. on Z.*

REMARK. Here $\varphi_a : X \to \Pi_{i \in a} Y_i$ where $Y_i = \varphi_i[X]$ is defined as in Definition 3.3. No topology is assumed in this lemma.

PROOF. Let f be any element of $B(Z)$. Let $\hat{f} \in B(X)$ be such that $\hat{f}/Z = f$ and $\hat{f}/L = 0$. Since F is a u.s.f. on X, $\hat{f}$ is representable as $\hat{f}(x) = \Sigma_{i \in a} g_i(\varphi_i(x)) + \Sigma_{i \in b} g_i(\varphi_i(x))$, $g_i \in B(Y_i)$, $1 \leqq i \leqq k$.

φ_a is constant on L, i.e. $\varphi_a[L] = (y_{i_1}, y_{i_2}, \cdots, y_{i_{|a|}})$ and we may assume without loss of generality that $g_i(y_{i_j}) = 0$ for $i_j \in a$. Hence, for $x \in L$ we have:

$$0 = \hat{f}(x) = \sum_{i \in a} g_i(\varphi_i(x)) + \sum_{i \in b} g_i(\varphi_i(x)) = \sum_{i \in b} g_i(\varphi_i(x)).$$

Since $Z \subset \varphi_b^{-1}(\varphi_b[L])$, to each $z \in Z$ there correspond some $x \in L$ with $\varphi_b(z) = \varphi_b(x)$. Hence $\Sigma_{i \in b} g_i(\varphi_i(z)) = 0$ for all $z \in Z$. Thus, for $z \in Z$ we get

$$f(z) = \hat{f}(z) = \sum_{i \in a} g_i(\varphi_i(z)) + \sum_{i \in b} g_i(\varphi_i(z)) = \sum_{i \in a} g_i(\varphi_i(z))$$

and by Theorem 2.8, $\{\varphi_i\}_{i \in a}$ is a u.s.f. on Z. □

5.4. LEMMA. *Let X be an n-dimensional Cantor manifold $(n \geqq 2)$ and let $\{\varphi_i\}_{i=1}^k \subset C(X)$ be a minimal u.s.f. Then for each $a \subset [k]$, $|a| = k - 1$, $\dim \varphi_a = 0$ (see (4.16)).*

PROOF. We may assume that $a = [k-1]$. If $\dim \varphi_a > 0$ then there exists some point $\alpha = (\alpha_1, \cdots, \alpha_{k-1})$ in R^{k-1} with $\dim \varphi_a^{-1}(\alpha) \geqq 1$. Set $L = \varphi_a^{-1}(\alpha)$. Since $\{\varphi_i\}_{i \leqq k}$ is a u.s.f. and $\{\varphi_i\}_{i \leqq k-1}$ are constant on L, it follows that φ_k is one to one on

L. Hence φ_k/L is a homeomorphism, and L and $\varphi_k[L]$ are one dimensional. Thus $\varphi_k[L]$ has a non-empty interior in the line, and $\varphi_k^{-1}(\varphi_k[L])$ and also $\varphi_k^{-1}(\varphi_k[L])\setminus L$ have non-empty interior in X. Hence there exists an n-dimensional Cantor manifold $Z\subset\varphi_k^{-1}(\varphi_k[L])\setminus L$. By Lemma 5.3, $\{\varphi_i\}_{i\in a}$ is a u.s.f. on Z — contradicting the minimality of $\{\varphi_i\}_{i=1}^k$.

We turn now to the case $n=2$.

5.5. LEMMA. *Let X be a two-dimensional compact metric space and let $F=\{\varphi_1,\varphi_2\}\subset C(X)$. Then F is not a u.s.f.*

PROOF. If F is a u.s.f., then $\varphi_{12}:X\to R^2$ is a homeomorphism. Hence $\dim\varphi_{12}[X]=2$ in R^2, and it follows that $\varphi_{12}[X]$ contains a square whose vertices are $(\alpha_1,\beta_1)(\alpha_2,\beta_2)(\alpha_2,\beta_1)$ and (α_1,β_2). Let x_1,x_2,z_1,z_2 be points of X so that $\varphi_{12}(x_1)=(\alpha_1,\beta_1)$, $\varphi_{12}(x_2)=(\alpha_2,\beta_2)$, $\varphi_{12}(z_1)=(\alpha_2,\beta_1)$ and $\varphi_{12}(z_2)=(\alpha_1,\beta_2)$. Then $\varphi_i[\{x_1,x_2\}]=\varphi_i[\{z_1,z_2\}]$ for $i=1,2$, and F is not a u.s.f. □

5.6. COROLLARY. *Let X be a two-dimensional Cantor manifold and let $F=\{\varphi_i\}_{i=1}^3$ be a u.s.f., then for each $i,j\in[3]$ $\dim\varphi_{ij}=0$.*

PROOF. From Lemma 5.5 it follows that F is a minimal u.s.f. Hence by Lemma 5.4, $\dim\varphi_{ij}=0$. □

5.7. COROLLARY. *With the notation of Corollary 5.6 each φ_{ij} is interior.*

PROOF. Since $\dim\varphi_{ij}=0$, we get by (4.18) that

$$\dim\varphi_{ij}[U]\geqq\dim U-\dim\varphi_{ij}=\dim U$$

holds for all $U\subset X$. In particular, if $U\subset X$ is open, then $\dim U=2$ hence $\dim\varphi_{ij}U\geqq\dim U=2$ in the plane, and it follows that $\varphi_{ij}[U]$ has non-empty interior.† □

5.8. LEMMA. *Let X be a two-dimensional compact metric space, and let $F=\{\varphi_i\}_{i=1}^3\subset C(X)$. Then F is not a u.s.f.*

PROOF. Assume F is a u.s.f. on X. Since X contains a two-dimensional Cantor manifold we may assume that X itself is such. By Corlllary 5.7 the mappings $\varphi_{12},\varphi_{13},\varphi_{23}$ are all interior. Consider the family $F'=\{\varphi_i\}_{i=1}^4$ where $\varphi_4=\varphi_1$. It follows that the triples $(1,2,3)$ and $(4,2,3)$ are interior w.r.t. F' (see Definition 3.4) (we write i instead of $\{i\}$) and the sequence of triples

† (4.18) is applicable for U, since U contains a compact subset V with $\dim U=\dim V$. The same remark applies to the proof of Lemma 5.9.

$$(1,2,3),\ (4,2,3),\ (1,2,3),\ (4,2,3),\cdots$$

is an interior chain w.r.t. F' of arbitrary length. By the main lemma F' is not a u.s.f., and hence neither F is such. □

5.9. LEMMA. *Let X be a two-dimensional Cantor manifold and let $F = \{\varphi_i\}_{i=1}^4 \subset C(X)$ be a u.s.f. There exist a permutation π of $[4]$ and a two-dimensional Cantor manifold $X' \subset X$ such that the restrictions to X' of the four functions*

$$\varphi_{\pi(4),\pi(3)},\ \varphi_{\pi(4),\pi(2)},\ \varphi_{\pi(1),\pi(3)},\ \varphi_{\pi(1),\pi(2)}$$

are interior on X'.

PROOF. By Lemma 5.8 F is minimal. Hence by Lemma 5.4, for all $a \subset [4]$, $|a| = 3$ and $\dim \varphi_a = 0$. By Theorem 4.19 we get that for all $U \subset X$ with $\operatorname{dc} U = 2$, $\operatorname{dc} \varphi_a[U] \geqq \operatorname{dc} U - \dim \varphi_a = 2$.

Let us see first that two functions of the form φ_{ij} with a common index i, e.g. $\varphi_{1,2}$ and $\varphi_{1,3}$, cannot both reduce the dimension of a two-dimensional subset of X. Indeed, assume $\dim \varphi_{12}[U] = \dim \varphi_{13}[U] = 1$ for some $U \subset X$ with $\dim U = 2$. Since each such U contains a two-dimensional Cantor manifold, we may assume that U is such (clearly $\dim \varphi_{ij}[U] = 0$ is impossible). As we remarked $\operatorname{dc} \varphi_{123}[U] \geqq 2$. The two-dimensional projections of $\varphi_{123}[U]$ in R^3 are $\varphi_{12}[U]$, $\varphi_{13}[U]$ and $\varphi_{23}[U]$, and by our assumption two of them, $\varphi_{12}[U]$ and $\varphi_{13}[U]$, are one dimensional. It is also clear that $\dim \varphi_1[U] = 1$ (otherwise φ_1 is constant on U and $\{\varphi_i\}_{i=2,3,4}$ is a u.s.f. on U contradicting Lemma 5.8). Hence we get a contradiction to Theorem 4.9.

We come now to the proof of the lemma: if all the functions φ_{ij} are interior on X then there is nothing to prove. Otherwise, one of them, say $\varphi_{2,3}$, is not interior, i.e. there exists a $U \subset X$ open such that $\operatorname{int} \varphi_{23}[U] = \emptyset$ in the plane. Let $X' \subset U$ be a two-dimensional Cantor manifold. Then, in particular $\dim \varphi_{23}[X'] \leqq 1$. From the above discussion it follows that for each function φ_{ij} with a common index with φ_{23}, and each $U \subset X'$ open $\dim \varphi_{ij}[U] = 2$ holds, i.e. the interior of $\varphi_{ij}[U]$ in R^2 is not empty. Hence the functions φ_{12}, φ_{13}, φ_{42}, φ_{43} are all interior. □

PROOF OF THEOREM 2.24 FOR $n = 2$. Let X be a two-dimensional compact metric space, and let $F = \{\varphi_i\}_{i=1}^4 \subset C(X)$. If F is a u.s.f., then by Lemma 5.9 we may assume that X is a two-dimensional Cantor manifold, and that φ_{12}, φ_{13}, φ_{24}, φ_{34} are all interior on X. Hence, the triples $(1,2,3)$ and $(4,2,3)$ are interior w.r.t. F and the sequence of triples

$$(1,2,3),\ (4,2,3),\ (1,2,3),\ (4,2,3),\cdots$$

is an interior chain w.r.t. F of arbitrary length. By the main lemma F is not a u.s.f. □

Before proving Theorem 2.24 for $n = 3$, we still need more information on the case $n = 2$.

5.10. Lemma. *Let X be a two-dimensional Cantor manifold, and let $F = \{\varphi_i\}_{i=1}^k \subset C(X)$ be a minimal u.s.f. Then for each $1 \leqq i \leqq k$ there exist $1 \leqq j_1 < j_2 \leqq k$, $j_1 \neq i \neq j_2$ so that* $\dim \varphi_{ij_1}[X] = \dim \varphi_{ij_2}[X] = 2$.

Proof. We may assume that $i = 1$. By Lemma 5.4 $\dim \varphi_{[k-1]} = 0$. Hence by Theorem 4.19 $\mathrm{dc}\, \varphi_{[k-1]}[X] \geqq 2$. Clearly φ_1 is not constant on X (otherwise $\{\varphi_i\}_{i=1}^k$ would not be minimal) and therefore $\dim \varphi_1[X] = 1$. Hence by Theorem 4.9 there exists some $j_1 \in [k-1]$, $j_1 \neq i$ so that $\dim \varphi_{i,j_1}[X] = 2$.

Set $a = ([k-1]\backslash\{j\}) \cup \{k\}$ then $|a| = k-1$, and again by Lemma 5.4 $\dim \varphi_a = 0$, and by Theorem 4.19 $\mathrm{dc}\, \varphi_a[X] \geqq 2$. Applying the same argument there exists some $j_2 \in a$, $j_2 \neq i$ so that $\dim \varphi_{i,j_2} = 2$. Clearly $j_1 \neq j_2$ and we are done. □

5.11. Lemma. *Let X be a compact metric two-dimensional space, and let $\{\varphi_i\}_{i=1}^k \subset C(X)$ be a u.s.f. on X. Then there exist $b, c \subset [K]$ with $b \cap c = \varnothing$ and $|b| = |c| = 2$ so that* $\dim \varphi_b[x] = \dim \varphi_c[x] = 2$.

Proof. By Lemma 5.2 we may assume that X is a two-dimensional Cantor manifold, and that $\{\varphi_i\}_{i=1}^k$ is a minimal u.s.f. for X. Let us call a pair $(i,j) \subset [k]$ a *good pair* if $\dim \varphi_{(i,j)}[x] = 2$. Clearly (by Theorem 4.8, for example) there exists a good pair. Assume that it is $(1,2)$. By Theorem 2.24 for $n = 2$, $k \geqq 5$. By Lemma 5.10 there exist $1 \leqq j_1 < j_2 \leqq k$ so that $(3, j_1)$ and $(3, j_2)$ are good pairs. If $j_1 \neq 1$ or $j_2 \neq 2$, then $(1,2)$ and $(3, j_1)$ or $(3, j_2)$ can be taken as b and c. If $j_1 = 1$ and $j_2 = 2$ then $(1,2)$, $(1,3)$ and $(2,3)$ are good pairs. By applying Lemma 5.10 again, there exist $1 \leqq j_1 < j_2 \leqq k$ so that $(4, j_1)$ and $(4, j_2)$ are good pairs. If $j_1 \neq 1$ or $j_2 \neq 2$ we can take $b = (1,2)$ and $c = (4, j_1)$ or $c = (4, j_2)$. If $j_1 = 1$ and $j_2 = 2$ then $(1,4)$ and $(2,4)$ are good pairs, and we may take $b = (1,3)$ and $c = (2,4)$. □

We come now to the case $n = 3$.

5.12. Lemma. *Let X be a compact three-dimensional metric space and let $\{\varphi_i\}_{i=1}^6 \subset C(X)$ be a u.s.f. for X. Then for each $1 \leqq i \leqq 6$ there exist $b, c \subset [6]$ with $|b| = |c| = 2$ and $b \cap c = \varnothing$ so that*

$$\dim \varphi_{\{i\}\cup c}[x] = \dim \varphi_{\{i\}\cup b}[x] = 3.$$

Proof. We may assume that X is a three-dimensional Cantor manifold.

Observe first that no family $F = \{\psi_i\}_{i=1}^5 \subset C(X)$ can be a u.s.f. for X. Indeed, by Theorem 4.17 there must be a $t \in \psi_5[X]$ with $\dim \psi_5^{-1}(t) \geqq 2$, and if $\{\psi_i\}_{i=1}^5$ is a u.s.f. for t, then $\{\psi_i\}_{i=1}^4$ is a u.s.f. for $\psi_5^{-1}(t)$ contradicting Theorem 2.24 for $n = 2$.

It follows that none of the φ_i's, $1 \leqq i \leqq 6$ is constant on X. Let us prove the lemma for $i = 6$. By the above remark $\varphi_6[X]$ is an interval $[\alpha, \beta]$. Each $\alpha < t < \beta$ disconnects $\varphi_6[X]$, hence $\varphi_6^{-1}(t)$ disconnects X. Since X is a three-dimensional Cantor manifold it follows that $\dim \varphi_6^{-1}(t) \geqq 2$. Clearly $\{\varphi_i\}_{i=1}^5$ is a u.s.f. for $\varphi_6^{-1}(t)$. By Lemma 5.11 there exist $b, c \subset [5]$ with $|b| = |c| = 2$ and $b \cap c = \varnothing$ so that $\dim \varphi_b[\varphi_6^{-1}(t)] = \dim \varphi_c[\varphi_6^{-1}(t)] = 2$. For each pair $b, c \subset [5]$ with $|b| = |c| = 2$ and $b \cap c = \varnothing$ set

$$T(b,c) = \{t \in \varphi_6[x] : \dim \varphi_b(\varphi_6^{-1}(t)) = \dim \varphi_c(\varphi_6^{-1}(t)) = 2\}. \tag{5.13}$$

It follows that

$$\bigcup_{b,c} T(b,c) \supset \{t : \alpha < t < \beta\}.$$

Hence there exists a pair b, c so that $T(b,c)$ is of second category in $[\alpha, \beta]$. The lemma now follows from Lemma (4.3) taking $X = \varphi_{\{6\}\cup b}[X]$ (resp $X = \varphi_{\{6\}\cup c}[X]$) and $Y = T(b,c)$. □

5.14. Lemma. *Let X and $\{\varphi_i\}_{i=1}^6$ be as in Lemma 5.12. Then there exists a $Y \subset X$ so that for each $1 \leqq i \leqq 6$ there exist $b, c \subset [6] \backslash \{i\}$ with $|b| = |c| = 2$ and $b \cap c = \varnothing$ such that $\varphi_{\{i\}\cup b}$ and $\varphi_{\{i\}\cup c}$ are interior on Y.*

Proof. We may assume that X is a three-dimensional Cantor manifold. Recall that for $Z \subset R^n$, $\operatorname{int} Z \neq \varnothing$ and $\dim Z = n$ are equivalent conditions.

Choose an index i, e.g. $i = 1$. If for all $b \subset \{2, 3, \cdots, 6\}$ with $|b| = 2$, and all open $U \subset X$ $\dim \varphi_{\{1\}\cup b}[U] = 3$ then we do nothing. If for some $b_1 \subset \{2, 3, \cdots, 6\}$, $|b_1| = 2$ and $U \subset X$ open $\dim \varphi_{\{1\}\cup b_1}[U] < 3$, we set $U = Y_1$. If there exist open $U \subset Y_1$ and another $b_2 \subset \{2, 3, \cdots, 6\}$, $|b_2| = 2$ with $\dim \varphi_{\{1\}\cup b_2}[U] < 3$, we take $U = Y_2$.

In this way we continue as far as we can. Suppose we end with b_k and Y_k. It follows that for all $1 \leqq j \leqq k$, $\dim \varphi_{\{1\}\cup b_j}[Y_k] < 3$. Let X_1 be a closed ball in Y_k. Then $\dim X_1 = 3$. Clearly $\{\varphi_i\}_{i=1}^6$ is a u.s.f. for X_1. By Lemma 5.12 there exist $b, c \subset \{2, 3, \cdots, 6\}$ with $|b| = |c| = 2$ and $b \cap c = \varnothing$ so that

$$\dim \varphi_{\{1\}\cup b}[X_1] = \dim \varphi_{\{1\}\cup c}[X_1] = 3.$$

Hence $b, c \not\in \{b_1 \cdots b_k\}$. If U is open in X_1 then its interior in Y_k is not empty. It follows that $\dim \varphi_{1\cup b}[U] = \dim \varphi_{1\cup c}[U] = 3$, because else we could take $U = Y_{k+1}$ and $b = b_{k+1}$ (or $c = b_{k+1}$), and continue the above procedure.

Now we take another index, e.g. $i = 2$, and operate on X_1 in the same manner with $i = 2$ to get X_2. Passing on all the six indices $i = 1, \cdots, 6$ we get X_6, and clearly X_6 can be taken to be Y. □

5.15. PROOF OF THEOREM 2.24 FOR $n = 3$. Let X be a three-dimensional compact metric space, and let $F = \{\varphi_i\}_{i=1}^6$ be functions in $C(X)$. If F is a u.s.f. on X, then by Lemma 5.14 there exists a closed $Y \subset X$ so that for each $1 \leqq i \leqq 6$ there exist $b, c \subset [6] \setminus \{i\}$, $|b| = |c| = 2$, $b \cap c = \emptyset$, and such that $\varphi_{i \cup b}$ and $\varphi_{i \cup c}$ are interior on Y. It follows that there exist interior chains w.r.t F of arbitrary length (relative to Y).

Indeed, choose any index $1 \leqq i \leqq 6$, e.g. $i = 1$. Set $a_1 = \{1\}$. Then there exist b_1, c_1 as above. (a_1, b_1, c_1) will be the first triple. Next set $a_2 = [6] \setminus a_1 \cup b_1 \cup c_1$ then $|a_2| = 1$, i.e. $a_2 = \{j\}$ for some $1 \leqq j \leqq 6$. Choose $b_2, c_2 \subset [6] \setminus a_2$ with $|b_2| = |c_2| = 2$, $b_2 \cap c_2 = \emptyset$ and so that $\varphi_{a_2 \cup b_2}$ and $\varphi_{a_2 \cup c_2}$ are interior (on Y). (a_2, b_2, c_2) will be the second triple in the chain. Setting $a_3 = [6] \setminus a_2 \cup b_2 \cup c_2$ we continue. By an obvious induction the chain can be continued as long as we please. Hence by the main lemma of Section 3, F is not a u.s.f. on Y, which is a contradiction. □

Let us turn to $n = 4$. First we prove a lemma of Lemma 5.12's type.

5.16. LEMMA. *Let X be a four-dimensional compact metric space, and let $\{\varphi_i\}_{i=1}^8 \subset C(X)$ be a u.s.f. Then for each $a \subset [8]$ with $|a| = 2$, there exists $b, c \subset [8] \setminus a$, $|b| = |c| = 2$, $b \cap c = \emptyset$ so that*

$$\dim \varphi_{a \cup b}[X] = \dim \varphi_{a \cup c}[X] = 4.$$

Let us first show how the case $n = 4$ of Theorem 2.24 follows from this lemma. The lemma will be proved later.

5.17. PROOF OF THEOREM 2.24 FOR $n = 4$. Let $\dim X = 4$, and $\{\varphi_i\}_{i=1}^8$ be a u.s.f. on X. First, using Lemma 5.16 we can show, in the same way as in the proof of Lemma 5.14 (replacing $i \in [6]$ by $a \subset [8]$), that there exist $Y \subset X$ such that for each $a \subset [8]$, $|a| = 2$, there exist $b, c \subset [8] \setminus a, |b| = |c| = 2$, $b \cap c = \emptyset$, so that $\varphi_{a \cup b}$ and $\varphi_{a \cup c}$ are interior on Y. It follows, as in the case $n = 3$ (with $|a_j| = 2$, instead of $|a_j| = 1$), that there exist interior chains of arbitrary length w.r.t. F and the theorem follows from the main lemma of Section 3. □

5.18. PROOF OF LEMMA 5.16. Let $a \subset [8]$ with $|a| = 2$. We claim first that $\dim \varphi_a = 2$. Indeed, by Theorem 4.17

$$\dim \varphi_a \geqq \dim X - \dim \varphi_a[X] \geqq 4 - 2 = 2.$$

On the other hand $\dim \varphi_a > 2$ would imply the existance of an $\alpha \in \varphi_a[X]$ so that $\dim \varphi_a^{-1}(\alpha) \geqq 3$, but the six functions $\{\varphi_i\}_{i \in [8] \setminus a}$ form a u.s.f. for $\varphi_a^{-1}(\alpha)$ contradicting Theorem 2.24 for $n = 3$. Hence $\dim \varphi_a = 2$.

Set

$$D_a = \{\alpha \in \varphi_a[X] : \dim \varphi_a^{-1}(\alpha) \geqq 2\}. \tag{5.19}$$

By Theorem 4.23 we get

$$\dim D_a \geqq \dim X - \dim \varphi_a = 2. \tag{5.20}$$

Hence D_a contains an open plane set.

For each $b, c \subset [8] \setminus a$, $|b| = |c| = 2$, $b \cap c = \varnothing$, set

$$T(b, c) = \{\alpha \in D_a : \dim \varphi_b[\varphi_a^{-1}(\alpha)] = \dim \varphi_c[\varphi_a^{-1}(\alpha)] = 2\}. \tag{5.21}$$

By Lemma 5.11 we get that

$$D_a \subset \bigcup_{b,c} T(b, c).$$

Hence there exist some b, c so that $T(b, c)$ is of second category in R^2. By Lemma 4.2 (with $X = \varphi_{a \cup b}[X]$ (resp. $X = \varphi_{a \cup c}[X]$), $Y = D_a$ and $k = 2$, $n = 4$) we get

$\dim \varphi_{a \cup b}[X] = \dim \varphi_{a \cup c}[X] = 4.$ □

REMARK. The natural approach to prove Theorem 2.24 for $n > 4$ is to use induction to show that if $\{\varphi_i\}_{i=1}^{2n}$ is a u.s.f. on an n-dimensional compact metric space X, then there exists a $1 \leqq k \leqq n$ so that for each $a \subset [2n]$ with $|a| = k$ there exist $b, c \subset [2n] \setminus a$ with $|b| = |c| = n - k$ and $b \cap c = \varnothing$ such that

$$\dim \varphi_{a \cup b}[X] = \dim \varphi_{a \cup c}[X] = n.$$

From such a situation one can continue as we did in the cases $n = 3, 4$. (Observe that we did the same, with $k = 1$ for $n = 3$ and $k = 2$ for $n = 4$.)

6. Proof of the main lemma

We shall prove the main lemma in its more precise setting 3.6; throughout this section we assume that X is a compact metric space, and that $F = \{\varphi_l\}_{l=1}^{2n} \subset C(X)$.

6.1. LEMMA. *If there exists an interior chain $\{(a_j, b_j, c_j)\}_{j=1}^{k}$ w.r.t. F, then there exist k^2 distinct points $\{x_i^j\}_{1 \leqq i,j \leqq k}$ in X such that for all $1 \leqq l \leqq 2n$, $\|T_l\mu\| < 8k$, where $\mu = \sum_{1 \leqq i,j \leqq k} (-1)^{i+j} \delta_{x_i^j} \in l_1(X)$. ($T$ is the operator defined in (2.3).)*

As observed in Remark 3.6, Lemma 6.1 implies the main lemma.

Before proving Lemma 6.1, let us examine an example which may clarify the nature of the points $\{x_i^j\}$ to be constructed in the proof of Lemma 6.1.

Let $\{(a_j, b_j, c_j)\}_{j=1}^k$ be an interior chain w.r.t. F. (The only properties of an interior chain which are relevant here are that $a_j \cup b_j \cup c_j \cup a_{j+1} = [2n]$, and that these sets are mutually disjoint).

Let $\{x_i^j\}$, $1 \leqq i, j \leqq k$ be k^2 distinct points in X so that

(i) $\varphi_{b_j}(x_i^j) = \varphi_{b_j}(x_{i+1}^j)$ for all $1 \leqq j \leqq k$ and i odd.

(ii) $\varphi_{c_j}(x_i^j) = \varphi_{c_j}(x_{i+1}^j)$ for all $1 \leqq j \leqq k$ and i even.

(iii) $\varphi_{a_j}(x_i^{j-1}) = \varphi_{a_j}(x_i^j)$ for all $1 \leqq j \leqq k$ and $1 \leqq i \leqq k$.

Figure 2 illustrates the situation. $x \overset{d}{—} y$ or $d \overset{x}{\underset{y}{|}}$ means $\varphi_d(x) = \varphi_d(y)$.

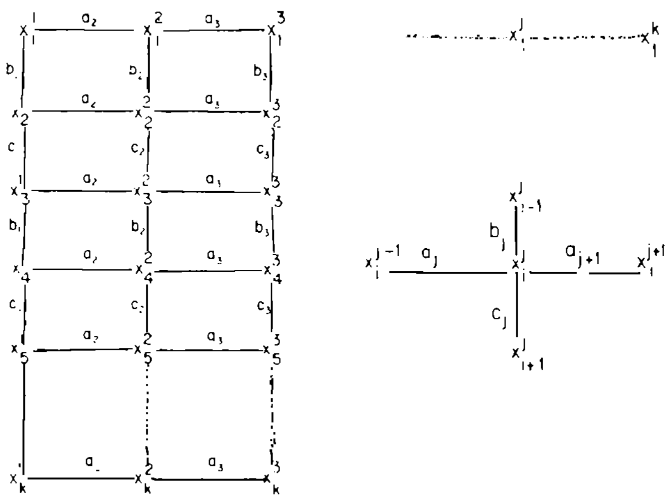

Fig. 2

We claim that if $\mu \in l_1(X)$ is defined by $\mu = \Sigma_{i,j}(-1)^{i+j}\delta_{x_i^j}$ then $\| T_l\mu \| < 4K$ for all $1 \leqq l \leqq 2n$. Indeed, fix an l, $1 \leqq l \leqq 2n$. Let x_i^j be an "interior" point of the "matrix" $A = \{x_i^j, 1 \leqq i, j \leqq k\}$, i.e. $1 < i, j < k$. Then l is an element in one and only one of the sets a_j, b_j, c_j, a_{j+1}. It follows that the correspondence $x_i^j \to \sigma_l(x_i^j)$ defined by

$$(*) \qquad \sigma_l(x_i^j) = \begin{cases} x_i^{j-1} & \text{if } l \in a_j \\ x_{i-(-1)^j}^j & \text{if } l \in b_j \\ x_{i+(-1)^j}^j & \text{if } l \in c_j \\ x_i^{j+1} & \text{if } l \in a_{j+1} \end{cases}$$

is well defined on the "interior" A' of A. It is also clear that if both x_i^j and $\sigma_l(x_i^j)$ are in A', then $\sigma_l^2(x_i^j) = x_i^j$. Hence σ_l can be extended to an idempotent permutation of $D_l = A' \cup \sigma_l[A']$. If $\sigma_l(x_i^j) = x_{i_0}^{j_0}$ for some $x_i^j \in D_l$, then by (*) we have that $(-1)^{i+j}\delta_{\varphi_l}(x_i^j) + (-1)^{i_0+j_0}\delta_{\varphi_l}(x_{i_0}^{j_0}) = 0$. Hence D_l can be decomposed into disjoint pairs, namely $\{x_i^j, \sigma_l(x_i^j)\}$, $x_i^j \in D_l$, so that the contribution of each pair to the norm of $T_l\mu$ is cancelled. It follows that the only points of A which may contribute to $\| T_l\mu \|$ are the points of $A \setminus D_l$, whose number is smaller than $4K$. Hence $\| T_l\mu \| < 4K$.

The points $\{x_i^j\}$ to be constructed in the proof of Lemma 6.1 will enjoy most of the properties of the points $\{x_i^j\}$ considered above i.e., that "most" of the points $\{x_i^j\}$ do not contribute to the norm of $T_l\mu$ for all $l \in [2n]$.

Before the construction we still need a definition and two lemmas whose proof will be brought after that of Lemma 6.1.

6.2. Definition. Let $a, b, c \subset [2n]$.

(i) A sequence $\{x_i\}_{i \leq k} \subset X$ will be called an alternating sequence w.r.t. b, c ($\langle b, c$ a.s.$\rangle$ in short) if

$$\varphi_b(x_i) = \varphi_b(x_{i+1}) \quad \text{for } i \text{ odd}$$

and

$$\varphi_c(x_i) = \varphi_c(x_{i+1}) \quad \text{for } i \text{ even}$$

holds.

(ii) A pair of sequences $\{x_i\}_{i=1}^m$, $\{x_i\}_{i=m+1}^{2m}$ will be called a doubly alternating sequence w.r.t. a; b, c ($\langle a; b, c$ d.a.s.$\rangle$ in short) if both $\{x_i\}_{i=1}^m$ and $\{x_i\}_{i=m+1}^{2m}$ are $\langle b, c$ a.s.$\rangle$ and, in addition, $\varphi_a(x_i) = \varphi_a(x_{2m+1-i})$ holds for $1 \leqq i \leqq m$. Figure 3

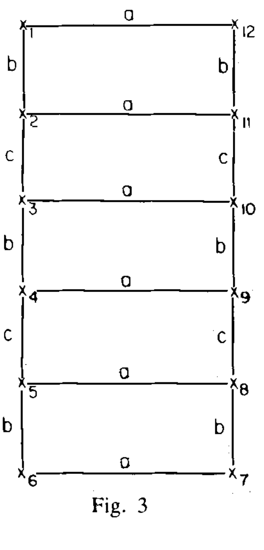

Fig. 3

illustrates the concepts of $\langle b, c$ a.s.$\rangle$ and $\langle a;\ b, c$ d.a.s.$\rangle$. $x_1 \overset{d}{-} x_2$ or $d \overset{x_1}{\underset{x_0}{|}}$ means $\varphi_d(x_1) = \varphi_d(x_2)$.

(iii) Let G', G be subsets of X, and let k be an integer. We use the notation $G' < G$ (rel a; b, c; k) to state that $G' \subset G$, and if $\{\alpha_i\}_{i=1}^k$ are points in $\varphi_a[G']$ so that $\alpha_i \neq \alpha_j$ for i even and j odd, then there exists a $\langle b, c$ a.s.$\rangle$ $\{x_i\}_{i=1}^k$ in G with $\varphi_a(x_i) = \alpha_i$, $1 \leqq i \leqq k$.

6.3. Lemma. *Let (a, b, c) be an interior triple w.r.t. F.*

(i) *For each integer $k > 0$, each open $G \subset X$ contains some open G' so that $G' < G$ (rel a; b, c; k).*

(ii) *Each open $G \subset X$ contains an $\langle a;\ b, c$ d.a.s.$\rangle$ of arbitrary length.*

6.4. Lemma. *Let $\{(a_j, b_j, c_j)\}_{j=1}^k$ be an interior chain w.r.t. F. Then there exist open sets $\{G_j\}_{j=1}^k$ and $\{G_j'\}_{j=1}^k$ in X so that the following holds:*

(i) $G_j' < G_j$ (rel a_j; b_j, c_j; k), $1 \leqq j \leqq k$.

(ii) $\varphi_{a_j}[G_{j-1}] \subset \varphi_{a_j}[G_j']$, $1 < j \leqq k$.

(iii) *The sets $\{G_j\}_{j=1}^k$ are mutually disjoint.*

We shall first prove Lemma 6.1, and then Lemma 6.3 and 6.4.

Proof of the Main Lemma. Let $\{(a_j, b_j, c_j)\}_{j=1}^k$ be an interior chain w.r.t. F. Let $\{G_j\}_{j=1}^k$ and $\{G_j'\}_{j=1}^k$ be open sets in X enjoying the properties (i), (ii), (iii) of Lemma 6.4. We shall construct the points $\{x_i^j\}$ so that for each $1 \leqq j \leqq k$ the points $\{x_i^j\}_{i=1}^k$ will be contained in G_j.

For $j = 1$, let $\{\alpha_i^1\}_{i=1}^k$ be k distinct points in $\varphi_{a_1}[G_1']$. Since $G_1' < G_1$ (rel a_1; b_1, c_1; k), there exists in G_1 a $\langle b_1, c_1$ a.s.$\rangle$ $\{x_i^1\}_{i=1}^k$ with $\varphi_{a_1}(x_i^1) = \alpha_i^1$, $1 \leqq i \leqq k$, hence the points $\{x_i^1\}_{i=1}^k$ are distinct.

For $j = 2$, set $\alpha_i^2 = \varphi_{a_2}(x_i^1)$, $1 \leqq i \leqq k$. By Lemma 6.4(ii) $\varphi_{a_2}[G_1] \subset \varphi_{a_2}[G_2']$, hence $\{\alpha_i^2\}_{i=1}^k \subset \varphi_{a_2}[G_2']$. Now we would like to apply the fact that $G_2' < G_2$ (rel a_2; $b_2 c_2$; k) to construct a $\langle b_2, c_2$ a.s.$\rangle$ $\{x_i^2\}_{i=1}^k$ in G_2 with $\varphi_{a_2}(x_i^2) = \alpha_i^2$; but to do this we have to ensure that $\alpha_{i_1}^2 \neq \alpha_{i_2}^2$ for i_1 even and i_2 odd (see Definition 6.2(iii)). In order to attain this, we remove from $\{\alpha_i^2\}_{i=1}^k$ a maximal number of pairs $\alpha_{i_1}^2 = \alpha_{i_2}^2$, i_1 even and i_2 odd.

In the set of remaining α_i^2's, α_i^2's with i even differs from α_i^2's with i odd. Assume that the number of removed pairs has been m_2. Then the number of remaining α_i^2's will be $r_2 = k - 2m_2$. Since we removed pairs of a_i^2's, one with i even and the other with i odd, we may assume without loss of generality that the remaining α_i^2's are $\{\alpha_i^2\}_{i=1}^{r_2}$. Hence, there exists a $\langle b_2, c_2$ a.s.$\rangle$ $\{x_i^2\}_{i=1}^{r_2}$ in G_2 with $\varphi_{a_2}(x_i^2) = \alpha_i^2$, $1 \leqq i \leqq r_2$. By Lemma 6.3(ii) we can choose in $G_2 \backslash \{x_i^2\}_{i=1}^{r_2}$ an $\langle a_2;\ b_2, c_2$ d.a.s.$\rangle$ $\{x_i^2\}_{i=r_2+1}^{r_2+m_2}$, $\{x_i^2\}_{i=r_2+m_2+1}^{k}$. This completes the construction of the points

$\{x_i^2\}_{i=1}^k$ in G_2. The points $\{x_i^j\}_{i=1}^k$ for $j=3,4,\cdots,k$ are constructed similarly. We may use induction: assume that the points $\{x_i^1\}_{i=1}^k\cdots\{x_i^{j_0-1}\}_{i=1}^k$ have been constructed with $\{x_i^j\}_{i=1}^k\subset G_j$, $1\leqq j\leqq j_0-1$. Set $\alpha_i^{j_0}=\varphi_{a_i}(x_i^{j_0-1})$. By Lemma 6.4(ii), $\{\alpha_i^{j_0}\}_{i=1}^k\subset\varphi_{a_{j_0}}[G'_{j_0}]$. Remove from the $\alpha_i^{j_0}$'s all the pairs of points $\alpha_{i_1}^{j_0}$, $\alpha_{i_2}^{j_0}$ with i_1 even, i_2 odd, and $\alpha_{i_1}^{j_0}=\alpha_{i_2}^{j_0}$, and assume that the remaining $\alpha_i^{j_0}$'s are $\{\alpha_i^{j_0}\}_{i=1}^{r_{j_0}}$ where $r_{j_0}=k-2m_{j_0}$, m_{j_0} being the number of removed pairs. Since $G'_{j_0}<G_{j_0}$ (rel a_{j_0}; b_{j_0},c_{j_0}; k) we can find a $\langle b_{j_0},c_{j_0}$ a.s.$\rangle$ $\{x_i^{j_0}\}_{i=1}^{r_{j_0}}$ in G_{j_0} with $\varphi_{a_{j_0}}(x_i^{j_0})=\alpha_i^{j_0}$, $1\leqq i\leqq r_{j_0}$. In $G_{j_0}\setminus\{x_i^{j_0}\}_{i=1}^{r_{j_0}}$ we pick an $\{a_{j_0};\ b_{j_0},c_{j_0}$ d.a.s.$\rangle$ $\{x_i^{j_0}\}_{i=r_{j_0}+1}^{r_{j_0}+m_{j_0}}$, $\{x_i^{j_0}\}_{i=r_{j_0}+m_{j_0}+1}^{k}$, and the points $\{x_i^{j_0}\}_{i=1}^k$ are constructed. By Lemma 6.4(iii), $\{x_i^{j_1}\}_{i=1}^k\cap\{x_i^{j_2}\}_{i=1}^k=\emptyset$ for $j_1\neq j_2$, and for a fixed j, $x_{i_1}^j\neq x_{i_2}^j$ for i_1 even and i_2 odd by our construction, hence $\mu=\Sigma_{i,j}(-1)^{i+j}\delta_{x_i^j}$ is a well defined element of norm k^2 in $l_1(X)$.

We claim that $\|T_l\mu\|<8K$ for all $1\leqq l\leqq 2n$. Indeed, fix an l, $1\leqq l\leqq 2n$. Set $A=\{x_i^j: 1\leqq i,\ j\leqq k\}$, and

$$A'=\{x_i^j: 1<j<k,\ i\not\in\{1,r_j,r_j+1,r_j+m_j,r_j+m_j+1,k\}\}.$$

As in the example, we shall construct an indempotent permutation σ_l on a set D_l, with $A'\subset D_l\subset A$, so that if $\sigma_l(x_i^j)=x_{i_0}^{j_0}$, then $(-1)^{i+j}+(-1)^{i_0+j_0}=0$, and $\varphi_l(x_i^j)=\varphi_l(x_{i_0}^{j_0})$.

It will follow, as in the example, that D_l can be decomposed into disjoint pairs (namely $\{x_i^j,\sigma_l(x_i^j)\}$, $x_i^j\in D_l$) so that the contribution of each pair to the norm of $T_l\mu$ is cancelled. Hence, the only points of A which may contribute to the norm of $T_l\mu$ are the points of $A\setminus D_l$. But $A\setminus D_l\subset A\setminus A'$ and $|A\setminus A'|\leqq k+k+6(k-2)<8K$. Thus $\|T_l\mu\|<8K$.

As in the example, we shall define σ_l on A' first, and then take $D_l=\sigma_l[A']\cup A'$. Recall that for each $1\leqq j<k$, $a_j\cup b_j\cup c_j\cup a_{j+1}=[2n]$, and that these sets are mutually disjoint.

Let $x_i^j\in A'$. Consider the following four cases:

1) $l\in b_j$. Then, if i is even, by our construction $\varphi_{b_j}(x_i^j)=\varphi_{b_j}(x_{i-1}^j)$ and we take $\sigma_l(x_i^j)=x_{i-1}^j$. If i is odd, then $\varphi_{b_j}(x_i^j)=\varphi_{b_j}(x_{i+1}^j)$ and we take $\sigma_l(x_i^j)=x_{i+1}^j$. Thus, if $l\in b_j$

$$\sigma_l(x_i^j)=\begin{cases}x_{i-1}^j & \text{if } i \text{ is even}\\ x_{i+1}^j & \text{if } i \text{ is odd.}\end{cases}$$

2) $l\in c_j$. Then similarly

$$\sigma_l(x_i^j)=\begin{cases}x_{i+1}^j & \text{if } i \text{ is even}\\ x_{i-1}^j & \text{if } i \text{ is odd.}\end{cases}$$

3) $l \in a_j$. If $1 < i < r_j$, then $\varphi_{a_j}(x_i^j) = \varphi_{a_j}(x_i^{j-1})$ and we take $\sigma_l(x_i^j) = x_i^{j-1}$. If $r_j + 1 < i < r_j + m_j$, then by the definition of an $\langle a_j, b_j, c_j$ d.a.s.$\rangle$ there exists some i', $r_j + m_j + 1 < i' < k$ in a congruence class mod 2 different from that of i (namely $i' = k + 1 + r_j - i$) so that $\varphi_{a_j}(x_i^j) = \varphi_{a_j}(x_{i'}^j)$ and we take $\sigma_l(x_i^j) = x_{i'}^j = x_{k+1+r_j-i}^j$.

A similar reasoning leads us to take $\sigma_l(x_i^j) = x_{i''}^j = x_{k+1+r_j-i}^j$ for $r_j + m_j + 1 < i < k$ (observe that $r_j + 1 < i'' < r_j + m_j$ in this case). Thus, if $l \in a_j$ we have

$$\sigma_l(x_i^j) = \begin{cases} x_i^{j-1} & \text{if } 1 < i < r_j \\ x_{k+1+r_j-i}^j & \text{if } i > r_j + 1. \end{cases}$$

4) $l \in a_{j+1}$. If $1 \leqq i \leqq r_{j+1}$ then $\varphi_{a_j}(x_i^j) = \varphi_{a_j}(x_i^{j+1})$ and we take $\sigma_l(x_i^j) = x_i^{j+1}$. If $i > r_j + 1$, then $\varphi_{a_j}(x_i^j) = \alpha_i^{j+1}$ is an element in one of the pairs which has been removed in the construction of the points of A (namely, of the points $\{x_i^{j+1}\}_{i=1}^k$). Hence there exists an i', $r_j < i' \leqq k$, with $i - i' \equiv 1 \pmod 2$ so that $\varphi_{a_{j+1}}(x_i^j) = \varphi_{a_{j+1}}(x_{i'}^j)$ and we take $\sigma_l(x_i^j) = x_{i'}^j$. (Observe that by the construction, the correspondence $i \to i'$ just defined is idempotent, i.e. $i \to i'$ implies $i' \to i$.)

This concludes the definition of σ_l on A'. It follows from the definition of σ_l that if both x_i^j and $\sigma_l(x_i^j)$ are in A', then $\sigma_l^2(x_i^j) = x_i^j$. Hence σ_l can be extended to an idempotent permutation of $D_l = A' \cup \sigma_l[A']$. It is also clear that if $\sigma_l(x_i^j) = x_{i_0}^{j_0}$ for some $x_i^j \in D_l$, then

$$(-1)^{i+j}\delta_{\varphi_l(x_i^j)} + (-1)^{i_0+j_0}\delta_{\varphi_l(x_{i_0}^{j_0})} = 0,$$

and the main lemma is proved. □

We still have to prove Lemmas 6.3 and 6.4.

Proof of Lemma 6.3. For $G, H \subset X$ and $d \subset [2n]$ we use the notation $H \subset_d G$ to state that $H = G \cap \varphi_d^{-1}(S)$ when S is a $|d|$ cube in $\varphi_d[G] \subset R^{|d|}$. (Recall that an m cube is an open cube in R^m with sides parallel to the axes.)

Let $\emptyset \neq G \subset X$ be an open set, let $k > 0$ be an integer, and let (a, b, c) be an interior triple w.r.t. F.

Set $G_k = G$. Since $\varphi_{a\cup b}$ is interior, there exists some n cube S in $\varphi_{a\cup b}[G_k]$ (recall that $|a \cup b| = n$). We may assume that $S = A \times B$ where A, B are $|a|$, $|b|$ cubes in $\varphi_a[G]$, $\varphi_b[G]$ resp.

Let B_k', B_k'' be $|b|$ cubes in B with $B_k' \cap B_k'' = \emptyset$. Set

$$S_k' = A \times B_k', \qquad S_k'' = A \times B_k''$$

(see [13, fig. 3, p. 307] where a similar construction has been carried out with $n = 2$, and $|a| = |b| = |c| = 1$) and

$$G'_k = G_k \cap \varphi^{-1}_{a\cup b}(S'_k), \qquad G''_k = G_k \cap \varphi^{-1}_{a\cup b}(S''_k).$$

Clearly $S'_k \cap S''_k = \emptyset = G'_k \cap G''_k$,

$$G'_k \subset_{a\cup b} G_k, \qquad G''_k \subset_{a\cup b} G_k$$

and G'_k, G''_k are open subsets of X.

Since $\varphi_{a\cup c}$ is interior, there exists an n cube $S'_{k-1} = A' \times C'$ in $\varphi_{a\cup c}[G'_k]$, where A', C' are $|a|$, $|c|$ cubes in $\varphi_a[G'_k]$, $\varphi_c[G'_k]$ respectively. Clearly $A' \subset A = \varphi_a[G'_k]$.

Set

$$G'_{k-1} = G'_k \cap \varphi^{-1}_{a\cup c}(S'_{k-1}).$$

It follows that $G = G_k \supset_{a\cup b} G'_k \supset_{a\cup c} G'_{k-1}$.

We return now to S''_k: $S''_k = A \times B''_k \supset A' \times B''_k$. Set $\hat{G}''_k = G''_k \cap \varphi^{-1}_{a\cup b}(A' \times B''_k)$. $\hat{G}''_k$ is open with $\varphi_a[\hat{G}''_k] = A'$.

Now $\varphi_{a\cup c}[\hat{G}''_k]$ contains an n cube $S''_{k-1} = A'' \times C''$ where $A'' \subset A'$. Set

$$G''_{k-1} = G''_k \cap \varphi^{-1}_{a\cup c}(S''_{k-1}).$$

Return again to S'_{k-1}. $S'_{k-1} = A' \times C' \supset A'' \times C'$. Set $\hat{G}'_{k-1} = \varphi^{-1}_{a\cup c}(A'' \times C') \cap G'_{k-1}$. $\varphi_{a\cup b}[\hat{G}'_{k-1}]$ contains an n cube $S'_{k-2} = A''' \times B'''$ with $A''' \subset A''$. Set

$$G'_{k-2} = G'_{k-1} \cap \varphi^{-1}_{a\cup c}(S'_{k-2}).$$

Then $G'_{k-1} \supset_{a\cup b} G'_{k-2}$.

Continuing in this manner we construct open sets $\{G'_i\}_{i=1}^k$ and $\{G''_i\}_{i=1}^k$ so that

$$G = G_k \supset_{a\cup b} G'_k \supset_{a\cup c} G'_{k-1} \supset_{a\cup b} G'_{k-2} \supset \cdots \supset_{a\cup c} G'_2 \supset_{a\cup b} G'_1,$$

$$G = G_k \supset_{a\cup b} G''_k \supset_{a\cup c} G''_{k-1} \supset_{a\cup b} G''_{k-2} \supset \cdots \supset_{a\cup c} G''_2 \supset_{a\cup b} G''_1$$

and

$$\varphi_a[G'_k] = \varphi_a[G''_k] \supset \varphi_a[G'_{k-1}] \supset \varphi_a[G''_{k-1}] \supset \varphi_a[G'_{k-2}] \supset \varphi_a[G''_{k-2}] \supset \cdots$$
$$\supset \varphi_a[G'_2] \supset \varphi_a[G''_1] \supset \varphi_a[G'_1] = A_1,$$

i.e. the $|a|$ cube A_1 is contained in all the $|a|$ cubes $\varphi_a[G'_i]$ and $\varphi_a[G''_i]$, $1 \leqq i \leqq k$.

We claim that $G'_1 < G$ (rel a; b, c; k). Indeed, let $\{\alpha_i\}_{i=1}^k$ be points in $\varphi_a[G'_1] = A_1$ such that α_i's with even i differ from α_i's with odd i. Let $\{z_i\}_{i=1}^k$ be points in G'_1 with $\varphi_a(z_i) = \alpha_i$, $1 \leqq i \leqq k$. Set $x_1 = z_1$. Since $G'_2 \supset_{a\cup b} G'_1$, and both x_1 and z_2 are in G'_1, it follows that there is a point x_2 in G'_2 with $\varphi_a(x_2) = \varphi_a(z_2) =$

α_2 and $\varphi_b(x_2) = \varphi_b(x_1)$. Let us explain this point: $G_1' = G_2' \cap \varphi_{a\cup b}^{-1}(S)$ where $S = A \times B$, $A \subset R^{|a|}$, $B \subset R^{|b|}$. Suppose the coordinates of $\varphi_{a\cup b}(x_1)$ and $\varphi_{a\cup b}(z_2)$ (in $R^{|a|} + R^{|b|}$) are $\varphi_{a\cup b}(x_1) = (\alpha_1, \beta_1)$ and $\varphi_{a\cup b}(z_2) = (\alpha_2, \beta_2)$. Since the cube $S = A \times B$ contains the points (α_1, β_1) and (α_2, β_2) it contains also the point (α_2, β_1), and we take as x_2 a point in G_2' for which $\varphi_{a\cup b}(x_2) = (\alpha_2, \beta_1)$.

Now we continue in a similar way: x_2 and z_3 are both in G_2', and $G_2' \supset_{a\cup c} G_3'$, thus as above there exists a point $x_3 \in G_3'$ with $\varphi_a(x_3) = \varphi_a(z_3) = \alpha_3$, and $\varphi_c(x_3) = \varphi_c(x_2)$, and by an obvious induction we construct the points $\{x_i\}_{i=1}^k$. An immediate check shows that $\varphi_a(x_i) = \alpha_i$, $1 \leqq i \leqq k$ and that $\{x_i\}_{i=1}^k$ is a $\langle b, c$ a.s.$\rangle$. Observe that $x_i \neq x_{i+1}$ (since the same is true for the α_i's) and that $\{x_i\}_{i \leqq k} \subset G_k'$. This proves (i) of Lemma 6.3. To prove Lemma 6.3(ii), construct in G_k'', in the same way, a $\langle b, c$, a.s.$\rangle$ $\{x_i\}_{i=k+1}^{2k}$, with $\alpha_i = \varphi_a(x_{2k+1-i})$, $1 \leqq i \leqq k$. This is possible since $\{\alpha_i\}_{i=1}^k \subset A_1 \subset \varphi_a[G_1'']$. It follows that the pair $\{x_i\}_{i=1}^k$ and $\{x_i\}_{i=k+1}^{2k}$ is a $\langle a; b, c$ d.a.s.$\rangle$. Observe that since $G_k' \cap G_k'' = \emptyset$, $\{x_i\}_{i=1}^k \cap \{x_i\}_{i=k+1}^{2k} = \emptyset$ too. Since k was arbitrary this proves Lemma 6.3(ii). □

Proof of Lemma 6.4. Set $X = X_k$. Since by the definition of an interior chain $\varphi_{a_k \cup b_k}$ is interior, it follows that $\varphi_{a_k \cup b_k}[X_k]$ contains an n cube $S = A \times B$, where A, B are $|a_k|$, $|b_k|$ cubes respectively. (Recall that by an m cube we mean an open m-dimensional cube in R^m whose sides are parallel to the axes of R^m, and that $|a_j \cup b_j| = |a_j \cup c_j| = n$ for $1 \leqq j \leqq k$.) Let B_1, B_2 be $|b_k|$ cubes in $R^{|b_k|}$ so that $B_1 \cap B_2 = \emptyset$, $B_1 \subset B$, $B_2 \subset B$ and set

$$G_k = X_k \cap \varphi_{a_k \cup b_k}^{-1}(A \times B_1).$$

Clearly G_k is open in X, hence by Lemma 6.3 there exists an open G_k' so that $G_k' < G_k$ (rel a_k; b_k, c_k; k). $\varphi_{a_k \cup b_k}[G_k']$ contains also some n cube $A' \times B'$ of the same type, and clearly $A' \subset A$.

Set $X_{k-1} = X_k \cap \varphi_{a_k \cup b_k}^{-1}(A' \times B_2)$, then X_{k-1} is open and

$$\varphi_{a_k}[X_{k-1}] = A' \subset \varphi_{a_k}[G_k']$$

and since $B_1 \cap B_2 = \emptyset$ it follows that $G_k \cap X_{k-1} = \varphi_{a_k \cup b_k}^{-1}(A \times B_1) \cap \varphi_{a_k \cup b_k}^{-1}(A' \times B_2) = \emptyset$ too.

Now operate in a similar manner on X_{k-1} with the interior triple $(a_{k-1}, b_{k-1}, c_{k-1})$, to construct in X_{k-1} open sets X_{k-2}, G_{k-1}, and G_{k-1}' so that

$$G_{k-1}' < G_{k-1} \text{ (rel } a_{k-1};\ b_{k-1}, c_{k-1};\ k),$$

$$G_{k-1} \cap X_{k-2} = \emptyset$$

and

$$\varphi_{a_{k-1}}[X_{k-2}] \subset \varphi_{a_{k-1}}[G'_{k-1}].$$

Since $G_{k-1} \subset X_{k-1}$, it follows that $G_k \cap G_{k-1} = \emptyset$.

By an obvious induction we continue, and construct open sets G_j and G'_j for $1 \leqq j \leqq k$ so that

(i) $G'_j < G_j$ (rel a_j; b_j, c_j; k) for $1 \leqq j \leqq k$,

(ii) $\varphi_{a_j}[G_{j-1}] \subset \varphi_{a_j}[G'_j]$ for $1 < j \leqq k$,

(iii) the sets $\{G_j\}_{j=1}^k$ are mutually disjoint.

And the lemma is proved. □

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